Generalization of the problem 1, IMO 2021, Day 1.

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(✠) If you find any English or math errors or if you have questions and/or suggestions, send me an email at **ilyasssabir7@gmail.com**

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Problem 1.

Let $n \geqslant 100$ be an integer. Ivan writes the numbers $n, n+1, \dots, 2n$ each on different cards. He then shuffles these $n+1$ cards, and divides them into two piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Solution.

We generalise this problem to the following question:

Generalization of the problem 1:

Let $r \in \mathbb{N}$ such that $r \geqslant 1$, and $n \geqslant \frac{5r^2}{2} + \frac{5r}{2}$ be an integer. Ivan writes the numbers $n, n+1, \dots, 2n$ each on different cards. He then suffles these $n+1$ cards, and divides them into $2r$ piles. Prove that at least one of the piles contains two cards such that the sum of their numbers is a perfect square.

Remark[1] that the **problem 1** is just a special case when $\boldsymbol{r = 1}$. And now we interess to solve this generalisation.

To have two cards in one of the piles such that the sum of their numers is a perfect square, it's suffisant to prove that there exist $r+1$ cards such that the sum of every pair of them is a perfect square.

1. In this case $(r = 1)$, we have $23r^2 + 19r + 3 = 45 < 100$.

Let $x_1, \ldots, x_{2r+1} \in \mathbb{N}$, and $a_1, \ldots, a_{2r+1} \in [\![n, 2n]\!]$ satisfy the following system of equations:

$$\begin{array}{rcl} a_1 + a_2 & = & x_1^2 \\ a_2 + a_3 & = & x_2^2 \\ & . & \\ & . & \\ & . & \\ a_{2r} + a_{2r+1} & = & x_{2r}^2 \\ a_{2r+1} + a_1 & = & x_{2r+1}^2 \end{array}$$

To conclude, it suffise to prove that this system of equations has a solution (with some conditions about $x_1, \ldots, x_n$), where the inconnues are $(a_1, \ldots, a_{r+1}) \in [\![n, 2n]\!]^{2r+1}$.

We have

$$\sum_{\text{cyc}} (a_1 + a_2) = \sum_{\text{cyc}} x_1^2 = \sum_{k=1}^{r+1} x_k^2$$

Notice

$$x_{2r+2} = x_1, x_{2r+3} = x_2, \ldots, x_{4r+2} = x_{2r+1}$$

and

$$a_{2r+2} = a_1, a_{2r+3} = a_2, \ldots, a_{4r+2} = a_{2r+1}$$

We have for all $l \in [\![0, r]\!]$

$$\begin{array}{rcl} \sum_{k=1}^{2r+1} (-1)^k (a_{k+l} + a_{k+l+1}) & = & \sum_{k=0}^{r-1} [(a_{2k+l+2} + a_{2k+l+3}) - (a_{2k+l+2} + a_{2k+l+1})] - (a_{2r+l+1} + a_{2r+2+l}) \\ & = & \sum_{k=0}^{r-1} [a_{2k+l+3} - a_{2k+l+1}] - (a_{2r+l+1} + a_{2r+2+l}) \\ & = & \sum_{k=0}^{r-1} [a_{2(k+1)+l+1} - a_{2k+l+1}] - (a_{2r+l+1} + a_{2r+2+l}) \\ & = & a_{2r+l+1} - a_{l+1} - (a_{2r+l+1} + a_{2r+2+l}) \\ & = & -a_{2r+2+l} - a_{l+1} \\ & = & -2a_{l+1} \end{array}$$

So for all $l \in [\![0, r]\!]$

$$a_{l+1} = \frac{1}{2} \sum_{k=1}^{2r+1} (-1)^{k-1} (a_{k+l} + a_{k+l+1}) = \frac{1}{2} \sum_{k=1}^{2r+1} (-1)^{k-1} x_{k+l}^2$$

By translation, we can conclude that for all $j \in [\![1, r+1]\!]$

$$a_j = \frac{1}{2} \sum_{k=1}^{2r+1} (-1)^{k-1} (a_{k+j-1} + a_{k+j}) = \frac{1}{2} \sum_{k=1}^{2r+1} (-1)^{k-1} x_{k+j-1}^2$$

We try to find $(x_1, x_2, \ldots, x_{2r+1}) \in \mathbb{N}$, satisfy $\frac{1}{2} \sum_{k=1}^{2r+1} (-1)^{k-1} x_{k+j-1}^2 \in [\![n, 2n]\!]$, for all $j \in [\![1, r+1]\!]$, where $x_{2r+2} = x_1, x_{2r+3} = x_2, \ldots, x_{4r+2} = x_{2r+1}$

To minimise the value of $\frac{1}{2} \sum_{k=1}^{2r+1} (-1)^{k-1} x_{k+j-1}^2$, for all $j \in [\![1, r+1]\!]$, it's normal to consider $x_1, x_2, \ldots,$ x_{2r+1} be secussives.

And because of the alternative sum, and to simplify the calcul, we consider $\alpha \in \mathbb{N}^\star$, such that for all $j \in [\![1, 2r+1]\!]$

$$x_j = \alpha + j - r$$

We have for all $j \in [\![1, 2r+1]\!]$

$$\begin{aligned} a_j &= \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}(\alpha + k + j - r - 1)^2 \\ &= \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}(\alpha^2 + 2(k + j - r - 1)\alpha + (k + j - r - 1)^2) \\ &= \left(\frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}\right)\alpha^2 + \left(\sum_{k=1}^{2r+1} (-1)^{k-1}(k + j - r - 1)\right)\alpha + \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}(k + j - r - 1)^2 \\ &= \frac{1}{2}\alpha^2 + \left(\sum_{k=1}^{2r+1} (-1)^{k-1}(k + j - r - 1)\right)\alpha + \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}(k + j - r - 1)^2 \end{aligned}$$

Were

$$\begin{aligned} \sum_{k=1}^{2r+1} (-1)^{k-1}(k + j - r - 1) &= \sum_{k=1}^{2r+1} (-1)^{k-1}k + (j - r - 1)\sum_{k=1}^{2r+1} (-1)^{k-1} \\ &= \sum_{k=1}^{r} [2k - (2k - 1)] + j - r - 1 + (2r + 1) \\ &= \left(\sum_{k=1}^{r} 1\right) + j + r \\ &= 2r + j \end{aligned}$$

And

$$\begin{aligned} \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}(k + j - r - 1)^2 &= \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}(k^2 + 2k(j - r - 1) + (j - r - 1)^2) \\ &= \frac{1}{2}\sum_{k=1}^{2r+1} (-1)^{k-1}k^2 + (j - r - 1)\sum_{k=1}^{2r+1} (-1)^{k-1}k + \frac{(j - r - 1)^2}{2}\sum_{k=1}^{2r+1} (-1)^{k-1} \\ &= \frac{1}{2}\sum_{k=1}^{r} [(2k)^2 - (2k - 1)^2] + (j - r - 1)r + \frac{(j - r - 1)^2}{2} + (2r + 1)^2 \\ &= \frac{1}{2}\sum_{k=1}^{r} [4k - 1] + (j - r - 1)r + \frac{(j - r - 1)^2}{2} + (2r + 1)^2 \\ &= r(r + 1) - \frac{r}{2} + (j - r - 1)r + \frac{(j - r - 1)^2}{2} + (2r + 1)^2 \\ &= \frac{9r^2}{2} + \frac{9r}{2} + \frac{j^2}{2} + \frac{3}{2} - j \end{aligned}$$

So

$$\begin{aligned} a_j &= \frac{1}{2}\alpha^2 + (2r + j)\alpha + \frac{9r^2}{2} + \frac{9r}{2} + \frac{j^2}{2} + \frac{3}{2} - j \\ &= \frac{1}{2}(\alpha - 1)^2 + (2r + j + 1)(\alpha - 1) + \frac{9r^2}{2} + \frac{13r}{2} + \frac{j^2}{2} + 2 \end{aligned}$$

Now, we interess to find the maximum and the minimum value of $a_1, \ldots, a_{2r+1}$. For that, we consider the function $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ definie as

$$\begin{aligned}\forall x \in \mathbb{R}, f(x) &= \frac{1}{2}\alpha^2 + (2r+x)\alpha + \frac{9r^2}{2} + \frac{9r}{2} + \frac{x^2}{2} + \frac{3}{2} - x \\ &= \frac{x^2}{2} + (\alpha-1)x + \frac{1}{2}\alpha^2 + 2r\alpha + \frac{9r^2}{2} + \frac{9r}{2} + \frac{3}{2}\end{aligned}$$

We have

$$\Delta_f = (\alpha-1)^2 - 2\left(\frac{1}{2}\alpha^2 + 2r\alpha + \frac{9r^2}{2} + \frac{9r}{2} + \frac{3}{2}\right) = -4\alpha(1+r) - 9r^9 - 9r - 2 < 0$$

And

$$\lim_{x \to +\infty} f(x) = +\infty$$

So f is increasing on $\mathbb{R}$.

Specifically

$$\min_{1 \leqslant j \leqslant 2r+1} a_j = a_1 \text{ and } \max_{1 \leqslant j \leqslant 2r+1} a_j = a_{2r+1}$$

With

$$a_1 = \frac{1}{2}\alpha^2 + (2r+1)\alpha + \frac{9r^2}{2} + \frac{9r}{2} + 1$$

And

$$a_{2r+1} = \frac{1}{2}\alpha^2 + (4r+1)\alpha + \frac{13r^2}{2} + \frac{9r}{2} + 1$$

The problem become to find $\alpha \in \mathbb{N}$, such that $a_1 \geqslant n$ and $a_{2r+1} \leqslant 2n$, so

$$\begin{aligned}\frac{1}{2}\alpha^2 + (2r+1)\alpha + \frac{9r^2}{2} + \frac{9r}{2} + 1 &\geqslant n \\ \frac{1}{2}\alpha^2 + (4r+1)\alpha + \frac{13r^2}{2} + \frac{9r}{2} + 1 &\leqslant 2n\end{aligned}$$

So because of $\alpha > 0$, we have :

$$\begin{aligned}\alpha &\geqslant -(2r+1) + \sqrt{2n - 5r^2 - 5r - 1} \\ \alpha &\leqslant -(4r+1) + \sqrt{4n + 3r^2 - r - 1}\end{aligned}$$

To have an integer between $r + \frac{1}{2} + \sqrt{2n - 5r^2 - 5r - 1}$ and $2r + \frac{1}{2} + \sqrt{2n + 3r^2 - r - 1}$, it suffisant that

$$\left(-(4r+1) + \sqrt{4n + 3r^2 - r - 1}\right) - \left(-(2r+1) + \sqrt{2n - 5r^2 - 5r - 1}\right) \geqslant 1$$

So

$$\sqrt{4n + 3r^2 - r - 1} - \sqrt{2n - 5r^2 - 5r - 1} - 2r - 1 \geqslant 0$$

We consider the function $h\colon \mathbb{R} \to \mathbb{R}$ definie as :

$$\forall x \geqslant \frac{5r^2}{2} + \frac{5r}{2}, h(x) = \sqrt{4x + 3r^2 - r - 1} - \sqrt{2x - 5r^2 - 5r - 1} - 2r - 1$$

The function h is derivable over the interval $\left]\frac{5}{2}r^2+\frac{5}{2}r+\frac{1}{2},+\infty\right[$. And for all $x\in\left]\frac{5}{2}r^2+\frac{5}{2}r+\frac{1}{2},+\infty\right[$ we have

$$\begin{aligned}
h'(x) &= \frac{2}{\sqrt{4x+3r^2-r-1}}-\frac{1}{\sqrt{2x-5r^2-5r-1}}\\
&= \frac{2\sqrt{2x-5r^2-5r-1}-\sqrt{4x+3r^2-r-1}}{\sqrt{(2x+3r^2-r-1)(2x-5r^2-5r-1)}}\\
&= \frac{4(2x-5r^2-5r-1)-(4x+3r^2-r-1)}{2\sqrt{2x-5r^2-5r-1}\sqrt{4x+3r^2-r-1}\left(2\sqrt{2x-5r^2-5r-1}+\sqrt{4x+3r^2-r-1}\right)}\\
&= \frac{4x-23r^2-19r-3}{\sqrt{(2x+3r^2-r-1)(2x-5r^2-5r-1)}\left(\sqrt{2x-5r^2-5r-1}+\sqrt{2x+3r^2-r-1}\right)}\\
&< 0
\end{aligned}$$

Therefore is increasing on $[23r^2+19r+3,+\infty[$

We have

$$\begin{aligned}
h(23r^2+19r+3) &= \sqrt{4(23r^2+19r+3)+3r^2-r-1}-\sqrt{2(23r^2+19r+3)-5r^2-5r-1}-2r-1\\
&= \sqrt{95r^2+75r+11}-\sqrt{41r^2+33r+5}-2r-1\\
&> 0 \qquad (1)
\end{aligned}$$

(1) : By simple study of the function $r\longmapsto\sqrt{95r^2+75r+11}-\sqrt{41r^2+33r+5}-2r-1$, we can show the positivity of $h(23r^2+19r+3)$

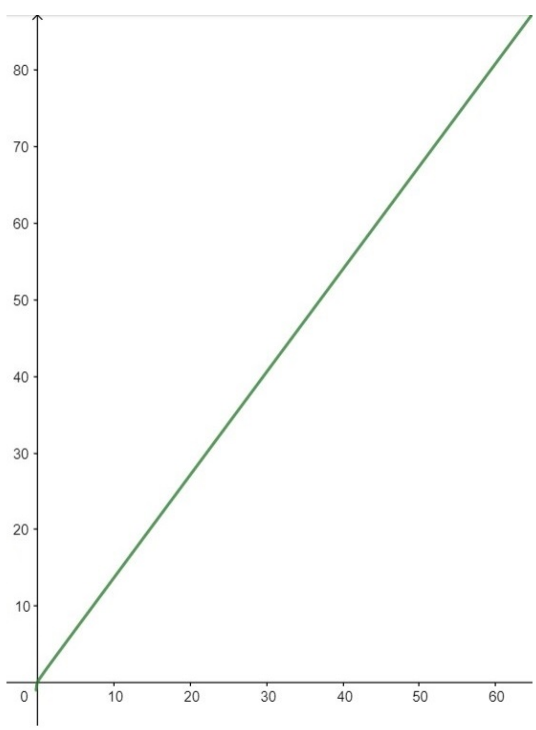

Figure 1. The graph of the function $r\longmapsto\sqrt{95r^2+75r+11}-\sqrt{41r^2+33r+5}-2r-1$

So for all $n \geqslant 23r^2 + 19r + 3$, we have

$$\left(-(4r+1)+\sqrt{4n+3r^2-r-1}\right)-\left(-(2r+1)+\sqrt{2n-5r^2-5r-1}\right)\geqslant 1$$

The problem is completly solved.

∇

Now, to go so far and make this problem more difficult, we offer some following challenges:

Some challenges

Challenge 1:

$\star$ Let $r > 1$, be an integer, find the minimum integer $f(r)$ such that for all $n \geqslant f(r)$, if we write the numbers $n, n+1, \ldots, 2n$ each on different cards. And we then shuffle these $n+1$ cards, and divides them into two piles. we have r cards in one of the piles such that the sum of their numbers is a perfect square.

Challenge 2:

$\star$ Let $r, s > 1$, be two integers, find the minimum integer $g(r, s)$ such that for all $n \geqslant g(r, s)$, if we write the numbers $n, n+1, \ldots, n+s$ each on different cards. And we then shuffle these $s+1$ cards, and divides them into two piles. we have r cards in one of the piles such that the sum of their numbers is a perfect square.

Challenge 3:

$\star$ Let $r, s, t, v, w > 1$, be five integers, find the minimum integer $h(r, s, t, v, w)$ such that for all $n \geqslant h(r, s, t, v, w)$, if we write the numbers $n, n+1, \ldots, n+s$ each on different cards. And we then shuffle these $s+1$ cards, and divides them into t piles. we have v cards in one of the piles such that the sum of their numbers is a perfect power w.